Signal sampling

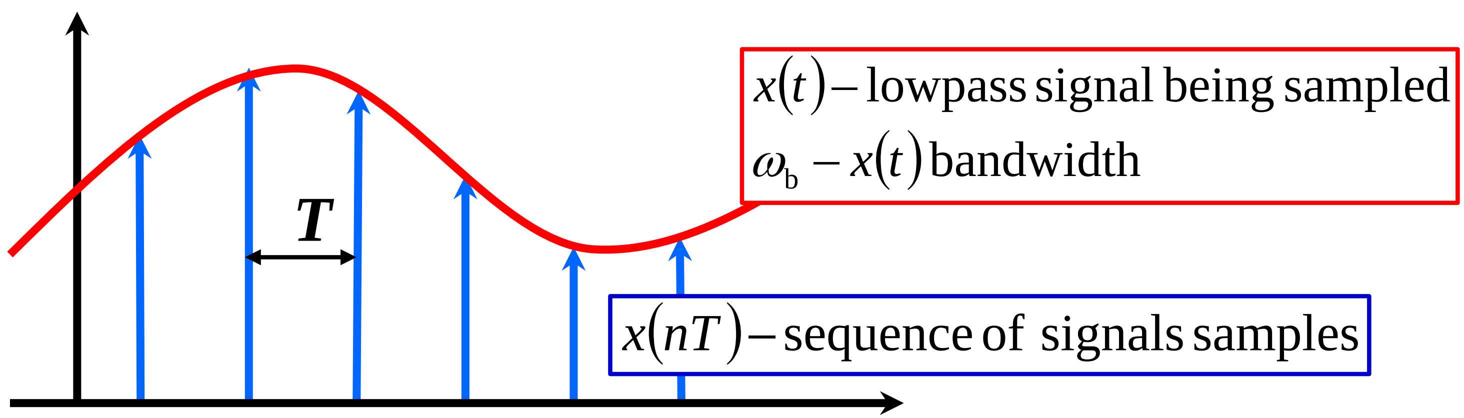
- Dirac delta – special Fourier transforms
- Dirac delta – sampling property
- Other properties of a Dirac delta
- Fourier transform pairs with delta pulse
- Impulse response of a filter revisited
- Comb delta
- Signal sampling with comb delta
- Sampling theorem
- Recovering signal from its samples
- Transmission of a sampled signal – a paradox
- Time Division Multiplex
- Summary

Decomposition of a signal

Exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \quad 0 \le t \le T \quad \omega_0 = 2\pi/T = 2\pi f_0$$

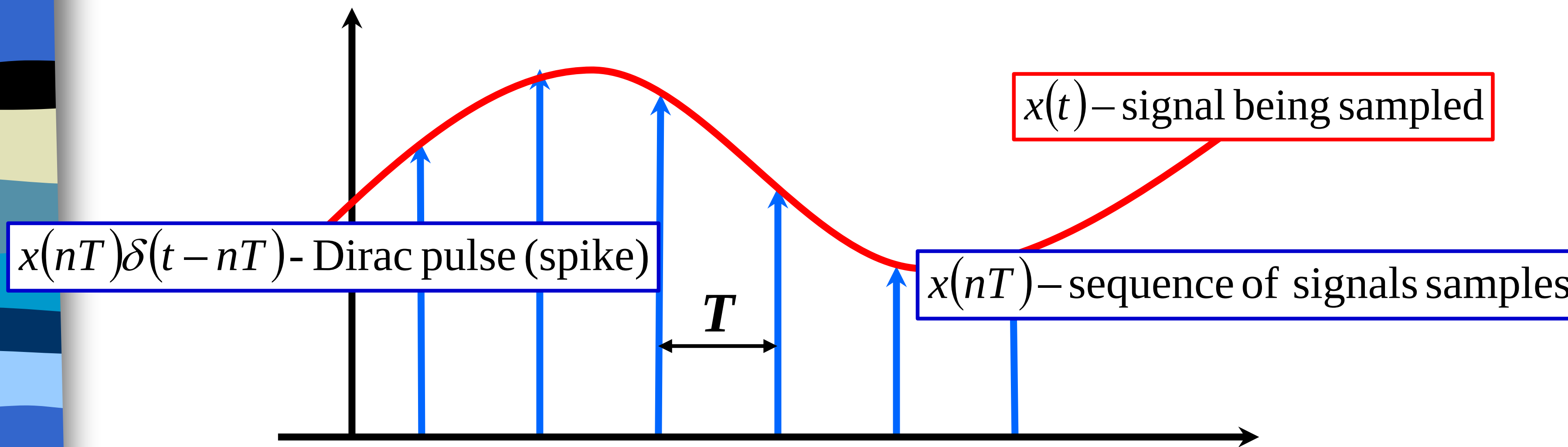
Maybe samples of a signal will be enough?



Kotielnikow – Shannon series (sampling theorem)

$$x(t) = \sum_{n=-\infty}^{n=\infty} x(nT)\,\mathrm{Sa}\,\omega_b(t - nT) \quad T = \pi/\omega_b$$

Dirac delta - sampling property



Dirac delta concept makes possible to analyze signal samples {$x(nT)$, n = 0, ± 1, ± 2…} in the spectral domain as a specific signal $x_s(t)$.
(other tools are Z-transform or Discrete Fourier Transform).

$$\{x(nT), n=0,\pm1,\pm2,\ldots\} \longrightarrow x_s(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t-nT) = x(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT)$$

$$x_s(t) \leftrightarrow \mathcal{F}\{x_s(t)\} = X_s(\omega)$$

Dirac delta – special Fourier transforms

$$x(t) = const \qquad x(t) = \cos\omega_0 t$$

$$x(t) = \mathbf{1}(t) \qquad x(t) = \sin\omega_0 t$$

$$x(t) = \operatorname{sgn}(t) \qquad x(t) = \mathbf{1}(t)\cos\omega_0 t$$

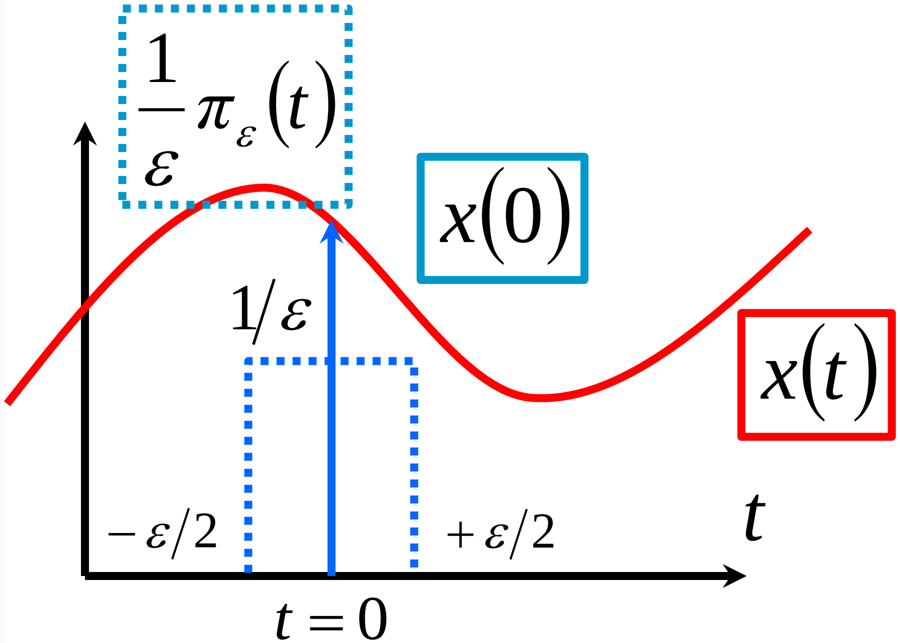
$$\text{Energy} = E = \int_{-\infty}^{\infty} x^2(t)\,dt = \infty$$

Signals $x(t)$ are not directly Fourier transformable because of the infinite energy E.

Signals $x(t)$ are often used in signal processing modeling, therefore, it would be recommended to find their Fourier transform (for its integrity).

Dirac delta concept makes possible to extend a class of Fourier transformable functions.

Dirac delta - sampling property



Difference quotient

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{1}{\varepsilon}\pi_\varepsilon(t)x(t)dt=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\varepsilon/2}^{+\varepsilon/2}x(t)dt=$$

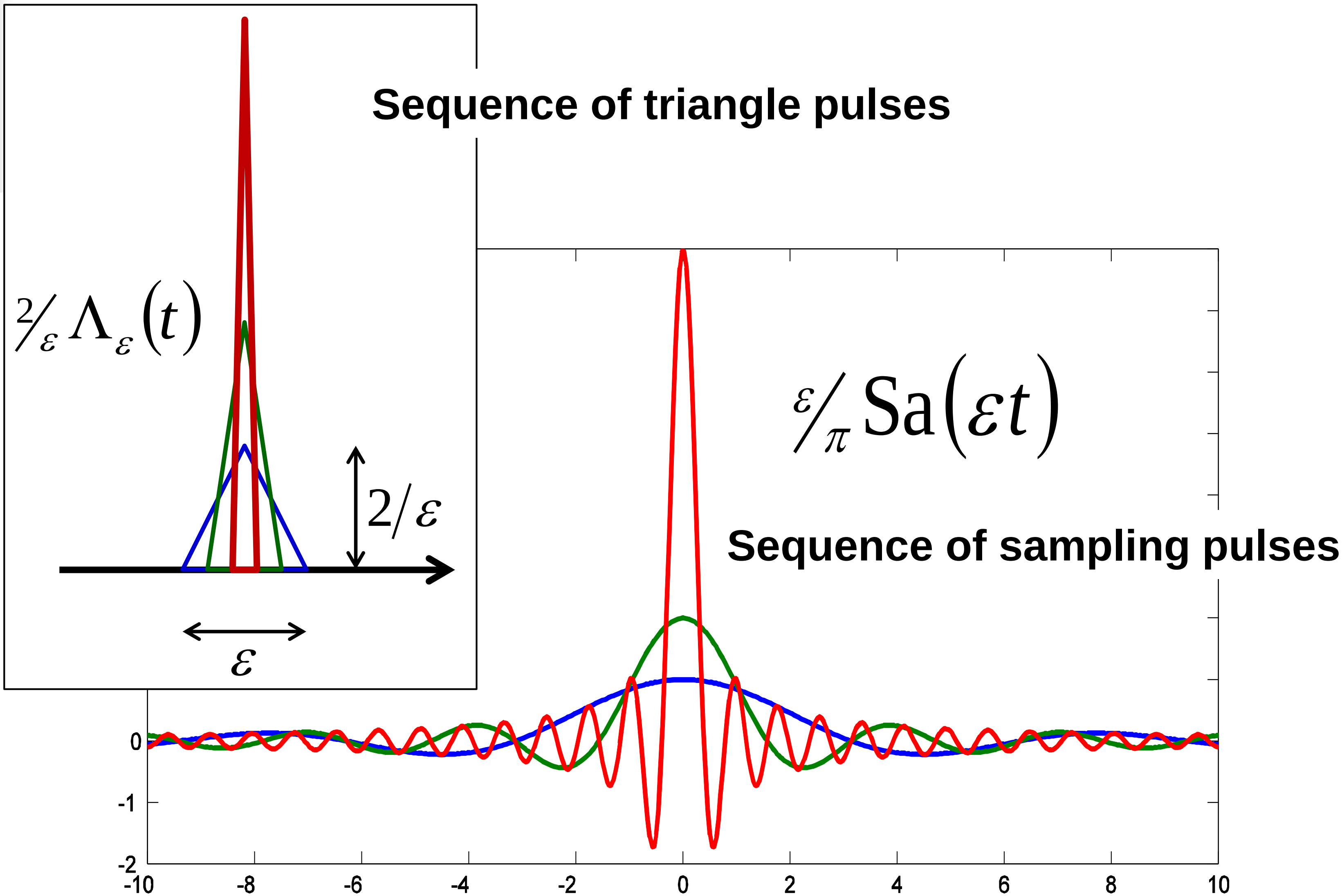
$X'(t)=x(t)$

indefinite integral

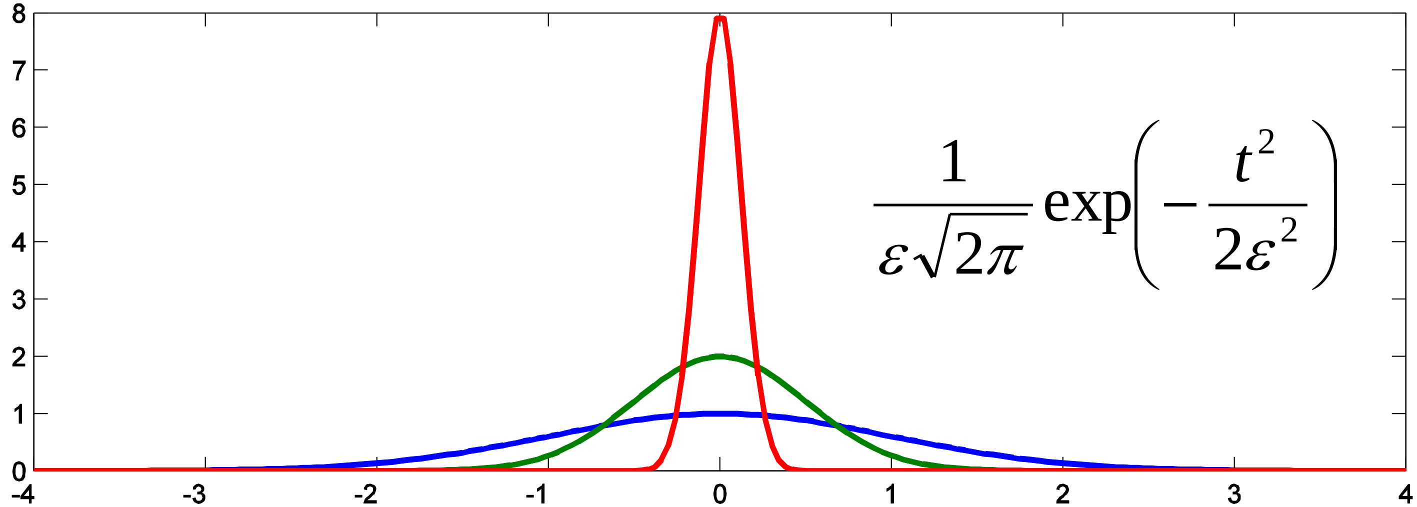
$$=\lim_{\varepsilon\to 0}\frac{X(+\varepsilon/2)-X(-\varepsilon/2)}{\varepsilon}=X'(0)=x(0)$$

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{1}{\varepsilon}\pi_\varepsilon(t)x(t)dt=\int_{-\infty}^{\infty}\underbrace{\left[\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\pi_\varepsilon(t)\right]}_{\delta(t)}x(t)dt=\underbrace{\int_{-\infty}^{\infty}\delta(t)x(t)dt=x(0)}_{\text{sampling property}}$$

Dirac delta (other pulse sequences)



Dirac delta (other pulse sequences)



$$\frac{1}{\varepsilon\sqrt{2\pi}}\exp\left(-\frac{t^2}{2\varepsilon^2}\right)$$

Sequence of Gaussian pulses

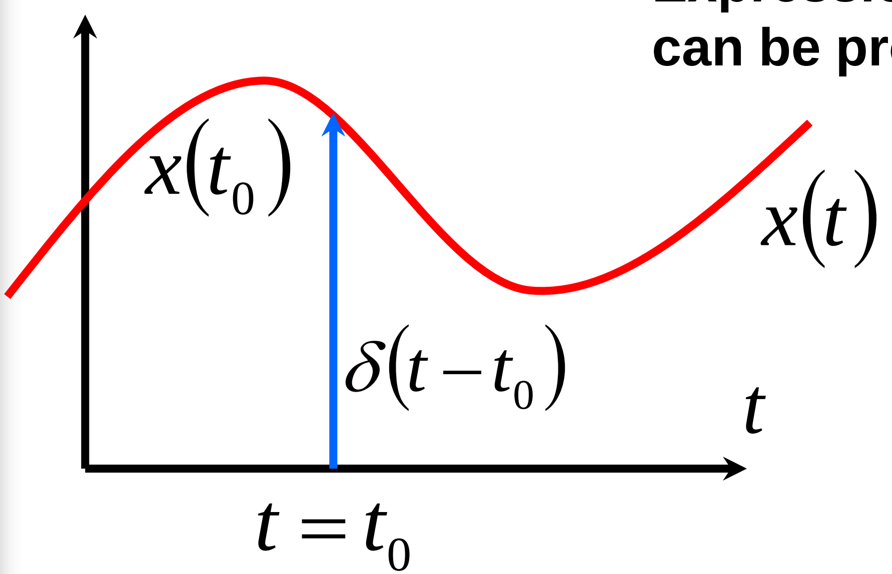
Dirac delta can be defined using
a sequence of pulses of any shape provided:
- pulse is getting higher as its width $\varepsilon \to 0$,
- pulse „area" is normalized to unity.

Sampling (sifting) property of a Dirac delta

$$x(t) \xrightarrow{\delta} \int_{-\infty}^{\infty} \delta(t) x(t) dt = x(0)$$

$$\int_{-\infty}^{\infty} \delta(t - t_0) x(t) dt = x(t_0)$$

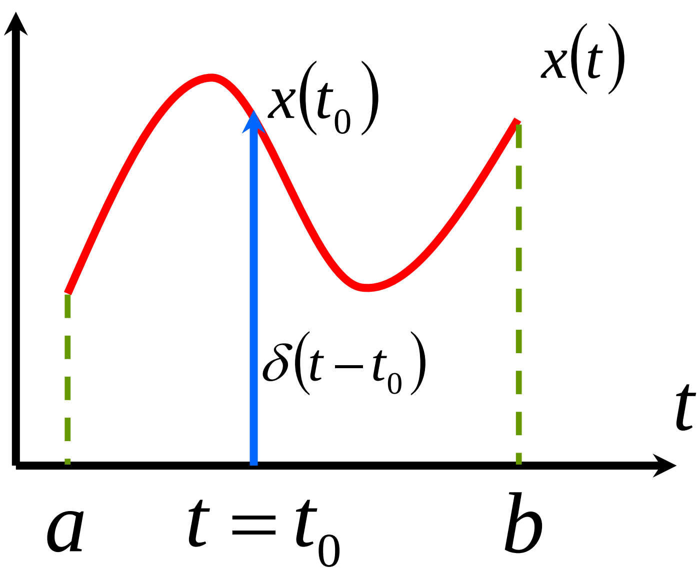
Expressions including Dirac delta can be processed in a common manner.



Sampling property of a Dirac delta

$$x(t) \xrightarrow{\delta} \int_{-\infty}^{\infty} \delta(t) x(t) dt = x(0)$$

$$\int_a^b \delta(t - t_0) x(t) dt = \begin{cases} x(t_0), t_0 \in [a, b] \\ 0, t_0 \notin [a, b] \end{cases}$$



Other properties of a Dirac delta

Dirac delta convolved

$$\delta(t) * x(t) = \int_{-\infty}^{\infty} \delta(\tau) x(t-\tau) d\tau = x(t)$$

$$\delta(t-t_0) * x(t) = x(t-t_0)$$

„Area” under a Dirac delta

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

Product with a Dirac delta

$$\int x(t)\delta(t-t_0)dt = x(t_0) = \int x(t_0)\delta(t-t_0)dt$$

$$\underbrace{x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)}_{\text{either transposition is correct}}$$

Dirac delta is an even function

$$\delta(-t) = \delta(t)$$

Dirac delta & unit step

$$\int_{-\infty}^{t} \delta(\tau) d\tau = \mathbf{1}(t)$$

Fourier transform pairs with delta pulse

$$\mathcal{F}\{\delta(t)\}=\int_{-\infty}^{\infty}\delta(t)\exp(-j\omega t)dt=1=const$$

Delta pulse

$$\delta(t)\leftrightarrow 1$$

Dc component

$$1\leftrightarrow 2\pi\delta(\omega)$$

Harmonic excitation (complex)

$$\exp(j\omega_0 t)=\cos\omega_0 t+j\sin\omega_0 t\leftrightarrow 2\pi\delta(\omega-\omega_0)$$

Unit step

$$\mathbf{1}(t)=\tfrac{1}{2}+\tfrac{1}{2}\operatorname{sgn}(t),\ \operatorname{sgn}(t)\leftrightarrow 2/(j\omega)$$

$$\mathbf{1}(t)\leftrightarrow \pi\delta(\omega)+1/(j\omega)$$

$$\mathcal{F}\{\operatorname{sgn}(t)\}=?\Rightarrow$$

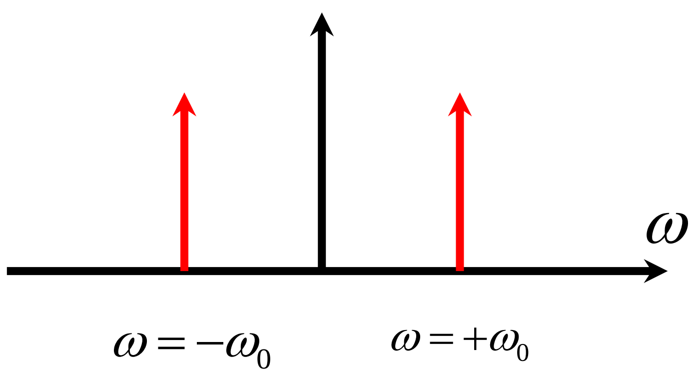
https://www.tutorialspoint.com/fourier-transform-of-signum-function

Fourier transform pairs with delta pulse

Harmonic signals

$$\left.\begin{aligned}\exp(+j\omega_0 t) &\leftrightarrow 2\pi\delta(\omega-\omega_0)\\ \exp(-j\omega_0 t) &\leftrightarrow 2\pi\delta(\omega+\omega_0)\end{aligned}\right\}\pm$$

$$\cos\omega_0 t \leftrightarrow \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$
$$\sin\omega_0 t \leftrightarrow -j\pi\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$$



Harmonic signals are represented in the frequency domain by a discrete line spectrum.

Impulse response of a filter revisited

$$\underset{X(j\omega)}{\xrightarrow{x(t)}} \boxed{H(s:=j\omega)} \underset{Y(j\omega)=H(j\omega)X(j\omega)}{\xrightarrow{y(t)}}$$

$$Y(j\omega)=H(j\omega)X(j\omega)$$
$$y(t)=h(t)*x(t)$$

$$h(t)\leftrightarrow H(j\omega)$$

impulse response

of a filter - why?

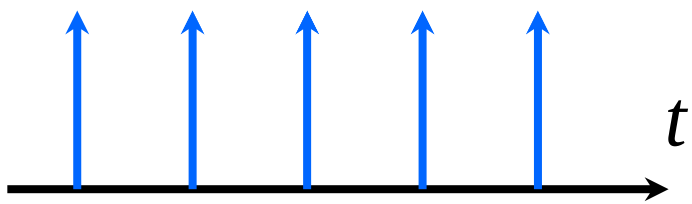
$$x(t)=\delta(t)\leftrightarrow X(j\omega)=1$$
$$Y(j\omega)=H(j\omega)$$
$$y(t)=h(t)$$
$$h(t)=h(t)*\delta(t)$$

Impulse response of a filter is equal to its output signal when a filter is excited by a Dirac delta pulse.

$$\xrightarrow{\delta(t)} \boxed{H(s:=j\omega)} \xrightarrow{y(t)=h(t)\leftrightarrow H(j\omega)}$$

Comb delta

$$\delta_T(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$$



$$\delta(t - nT)$$

Exponential Fourier series of the comb delta:

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) =$$
$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}, \omega_0 = 2\pi/T$$

Comb delta

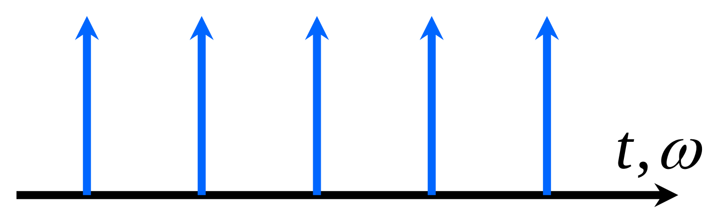
$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) =$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}, \omega_0 = 2\pi/T$$

$$\boxed{\delta_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \leftrightarrow \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)}$$

$$\boxed{\delta_T(t) \leftrightarrow \omega_0 \delta_{\omega_0}(\omega)}$$

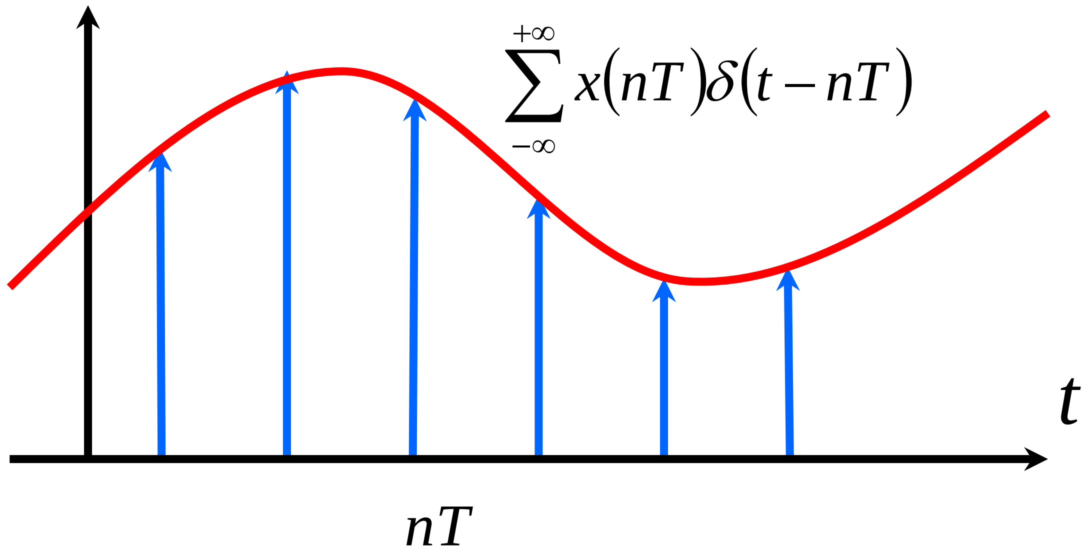
$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$\delta_{\omega_0}(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

Time comb delta and its Fourier transform are equal except to some constants.

Signal sampling with comb delta



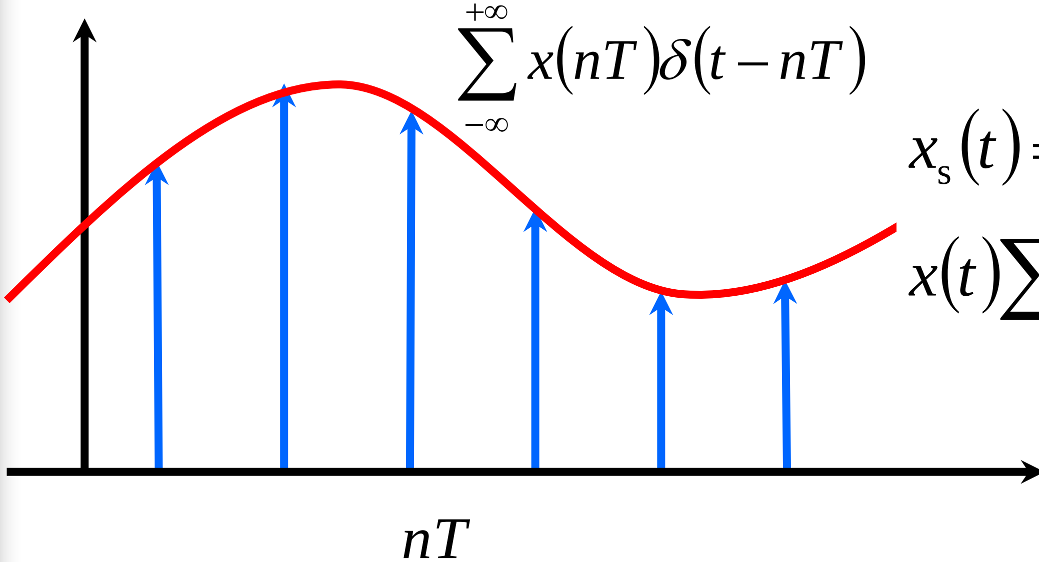
The comb delta models signal sampling:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) =$$

$$x(t)\sum_{n=-\infty}^{\infty} \delta(t-nT) = x(t)\delta_T(t)$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

Sampling theorem



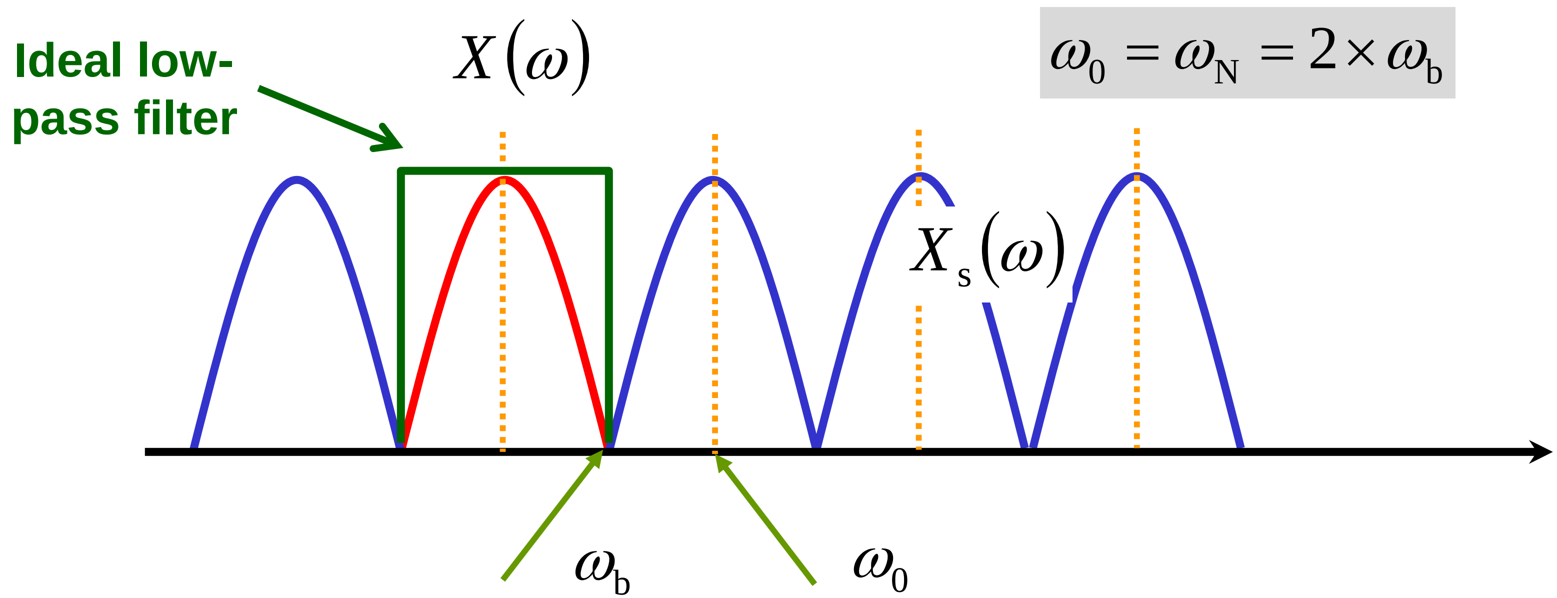
$$x_s(t)=\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)=$$
$$x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)=x(t)\delta_T(t)$$

$$X_s(\omega)=\mathcal{F}\{x_s(t)\}=\mathcal{F}\{x(t)\delta_T(t)\}=\frac{1}{2\pi}X(\omega)*\mathcal{F}\{\delta_T(t)\}=$$
$$=\frac{1}{2\pi}X(\omega)*\omega_0\delta_{\omega_0}(\omega)=\frac{1}{T}\sum_{n=-\infty}^{\infty}X(\omega-n\omega_0)$$
$$\omega_0=2\pi/T$$

$$\boxed{x_s(t)\leftrightarrow\frac{1}{T}\sum_{n=-\infty}^{\infty}X(\omega-n\omega_0)}$$

Critical (Nyquist) sampling

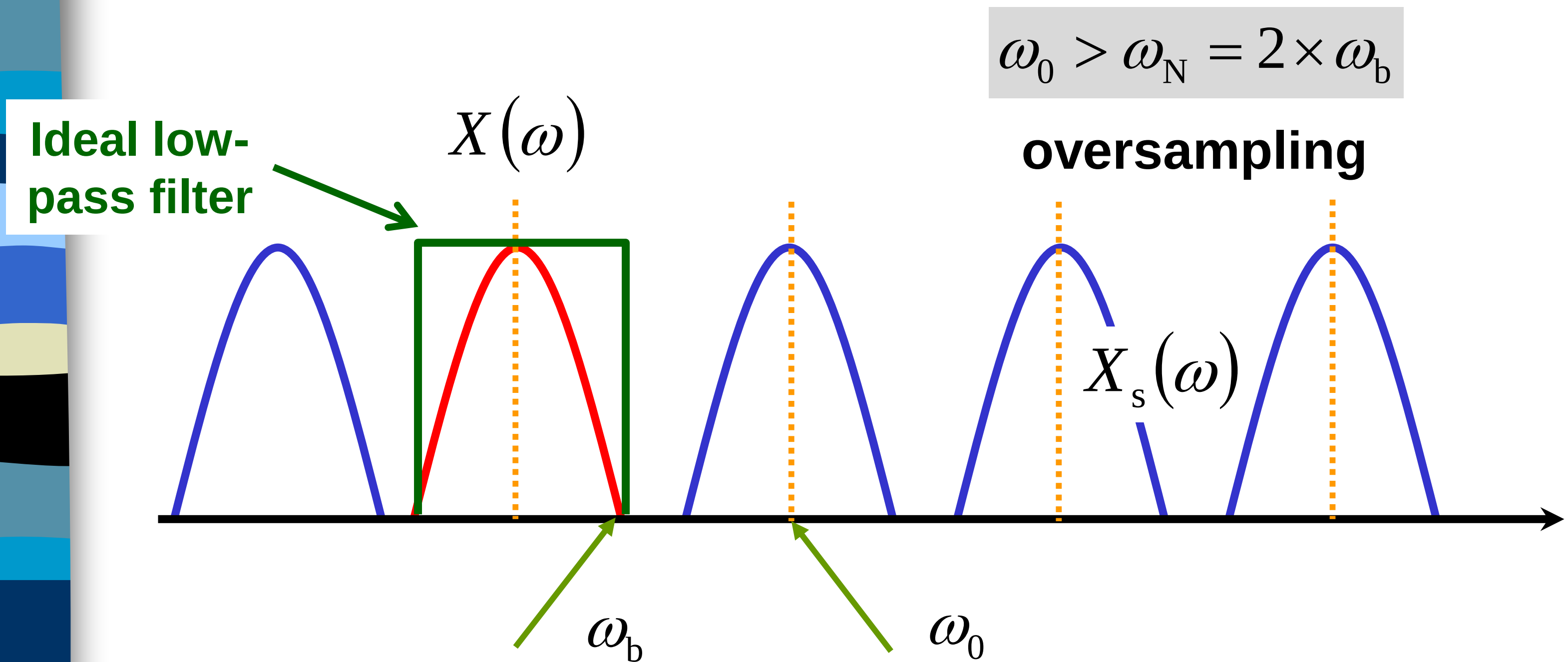
$$x_s(t) \leftrightarrow \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_0)$$



Sampling the lowpass signal at a frequency equal to the doubled signal bandwidth makes possible to recover signal from its samples via ideal lowpass filtering.

Oversampling

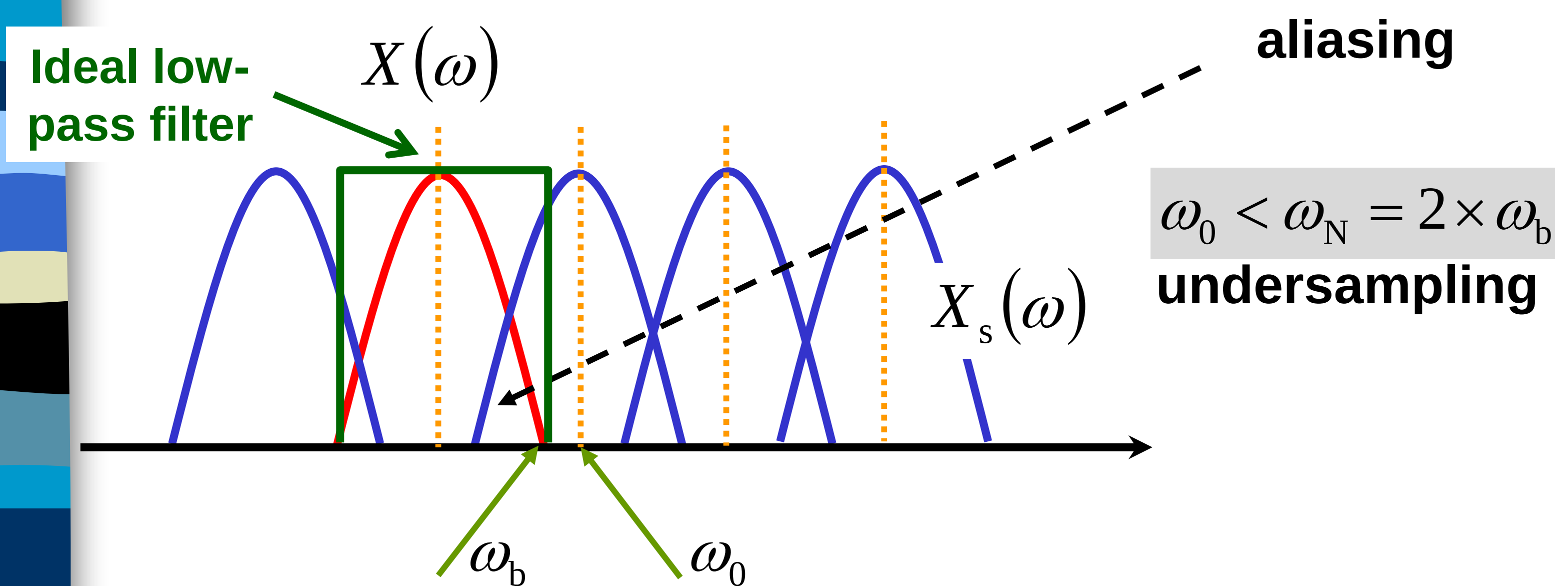
$$x_s(t) \leftrightarrow \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_0)$$



Sampling the lowpass signal at a frequency higher than the doubled signal bandwidth makes possible to recover signal from its samples (while providing frequency separation gaps of spectrum sidelobes).

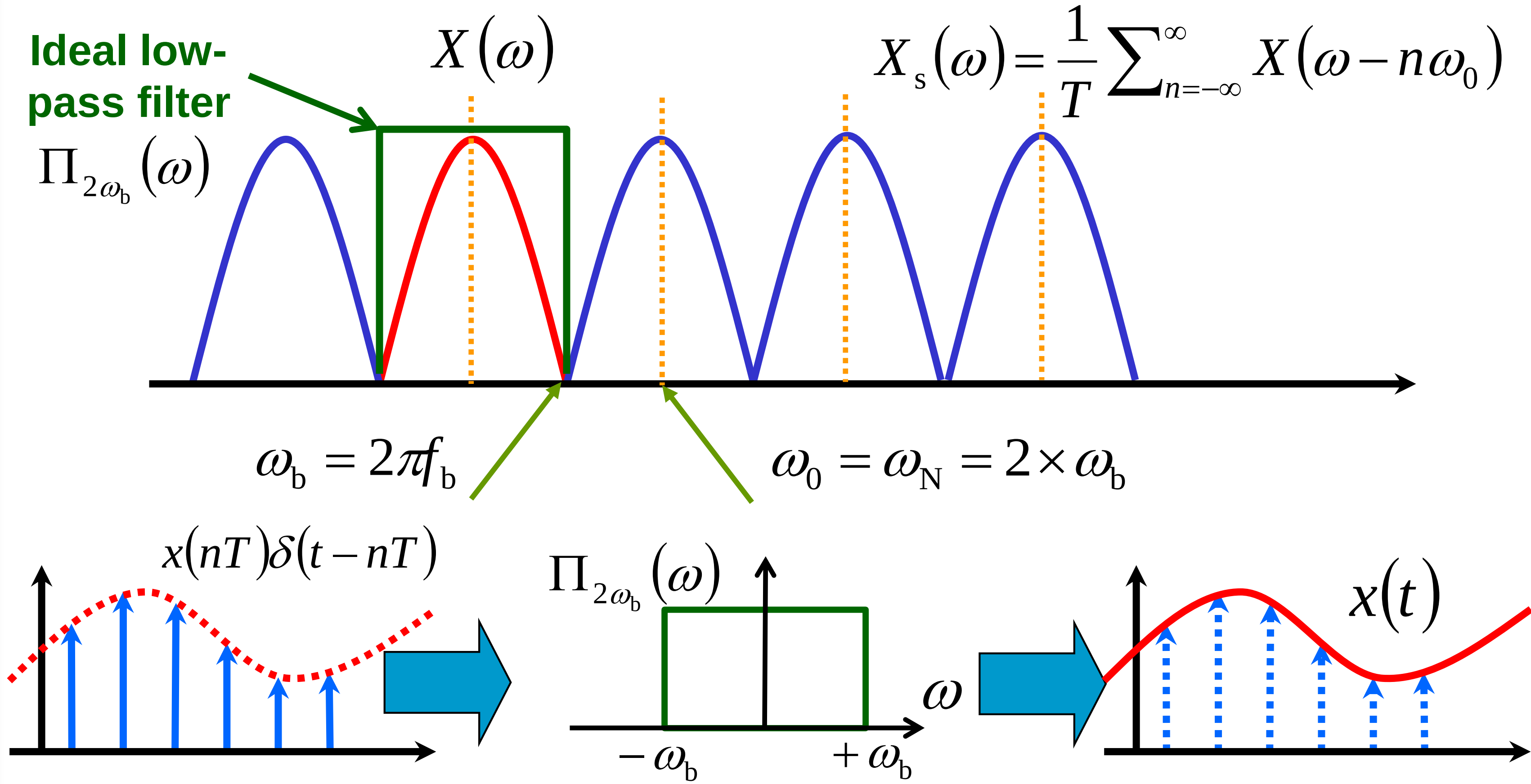
Undersampling

$$x_s(t) \leftrightarrow \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_0)$$



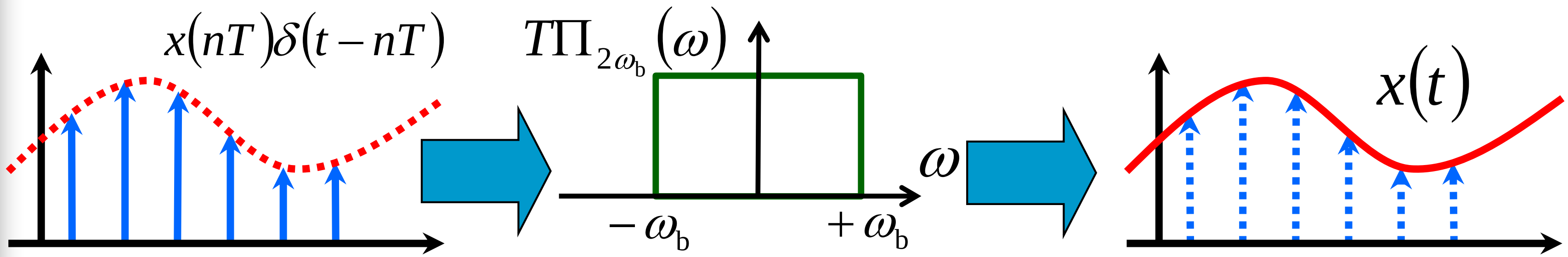
Sampling the lowpass signal at a frequency less than the doubled bandwidth introduces distortions because of overlapping of spectrum sidelobes (spectrum leakage, aliasing).

Recovering signal from its samples



The analogue signal (bandlimited to $\omega_b = 2\pi f_b$) can be exactly recovered from its samples (uniformly spaced in time with frequency $2f_b$) by passing the samples through an ideal lowpass filter with a bandwidth f_b.

Recovering signal from its samples



$$\mathrm{Sa}(Wt) \leftrightarrow \frac{\pi}{W}\Pi_{2W}(\omega)$$

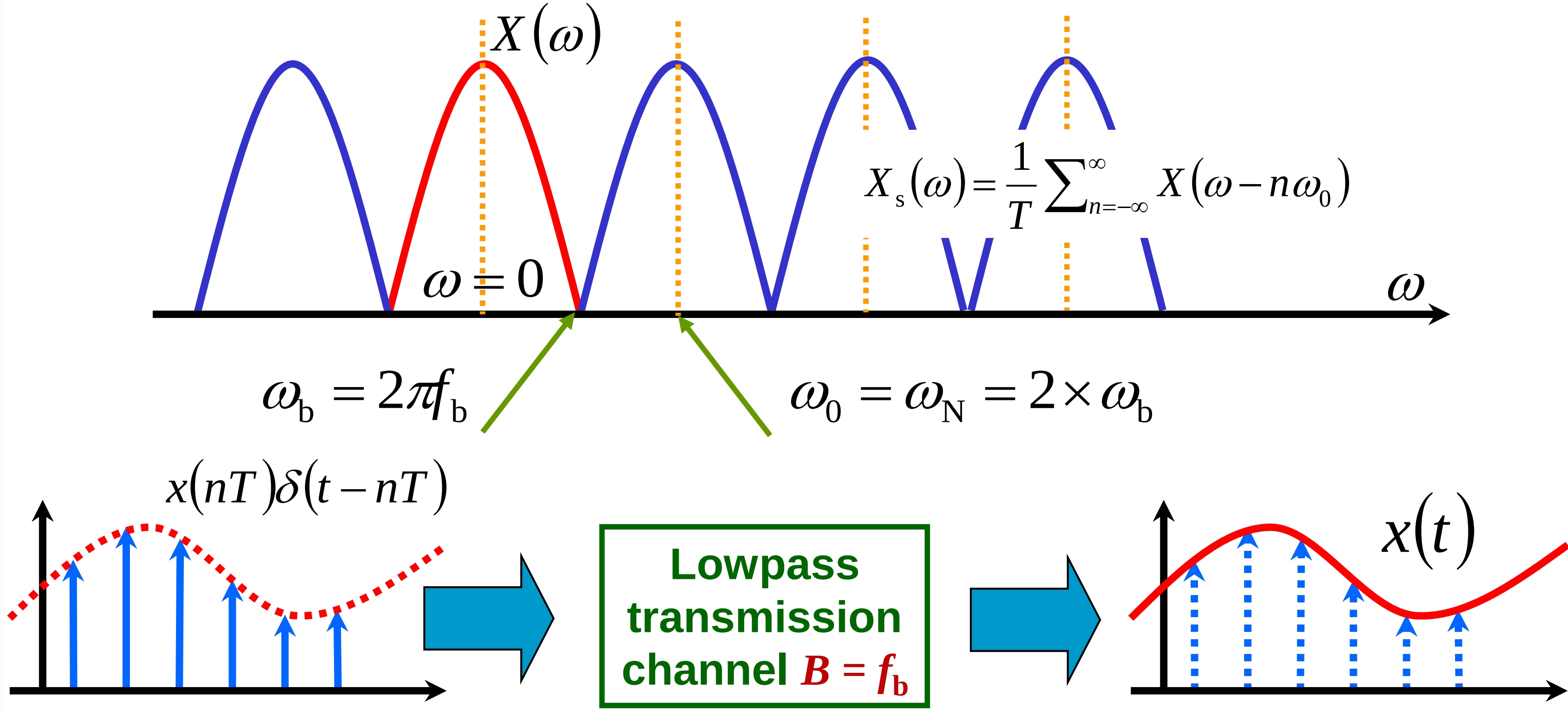
$$H(\omega) = T\Pi_{2\omega_b}(\omega) \leftrightarrow h(t) = \mathrm{Sa}(\pi t/T)$$

$$x_s(t) * h(t) = x(t) = \left[\sum_{n=-\infty}^{+\infty} x(nT)\delta(t-nT)\right] * \mathrm{Sa}(\pi t/T)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} x(nT)\mathrm{Sa}\frac{\pi(t-nT)}{T} = \sum_{n=-\infty}^{+\infty} x(nT)\mathrm{Sa}\,\omega_b(t-nT)$$

$$\omega_0 = \omega_N = 2\omega_b = \frac{2\pi}{T};\, \omega_b = \pi/T$$

Transmission of a sampled signal apparent paradox

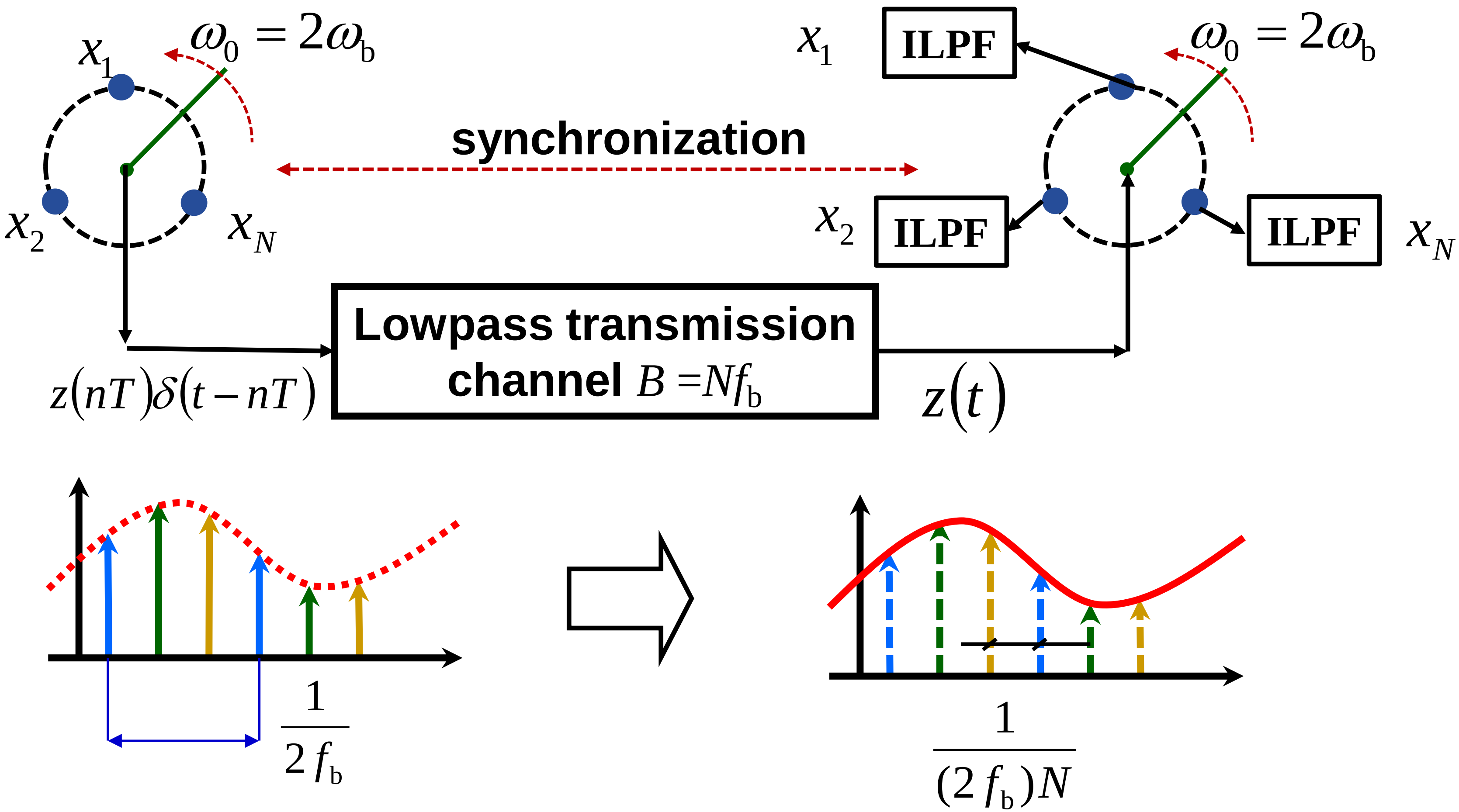


The minimum bandwidth of the lowpass transmission channel has to be equal to $W = \omega_b$ in order to avoid signal distortions.

However, it follows that the signal transmitted in the channel instead of being a train of equispaced samples is the original, analogue signal.

The paradox makes us doubting whether to use signal sampling for a <u>single signal</u> transmission.

TDM – Time Division Multiplexing



The signal transmitted in the channel is an analogue, composite si **$z(t)$ which equispaced samples are equal to samples of original sig** **$x_1, x_2, \ldots x_N$.**

The channel bandwidth necessary for transmission of the compos **analogue signal $z(t)$ is equal to $B = Nf_b$.**

Time Division Multiplexing - example

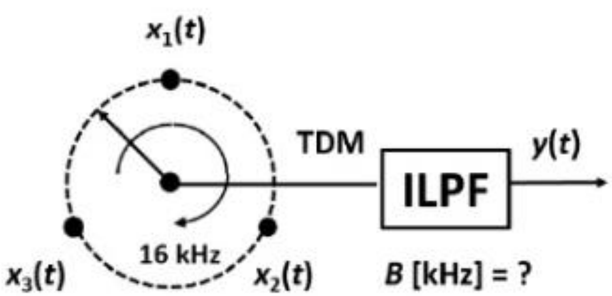
Bandwidth of two signals $x_1(t), x_2(t)$ is limited to 4 kHz each. The maximum frequency of the third signal $x_3(t)$ is equal to 8 kHz. The signals are sampled in the commutator as shown in the figure being converted to the TDM signal; the commutator frequency equals to 16 kHz. Lowpass filtration of the TDM samples in the ILPF produces the analogue signal $y(t)$.

Q1. Determine the frequency of samples in the TDM signal.

Q2. How much is the bandwidth B [kHz] of the filter ILPF and so the signal $y(t)$?

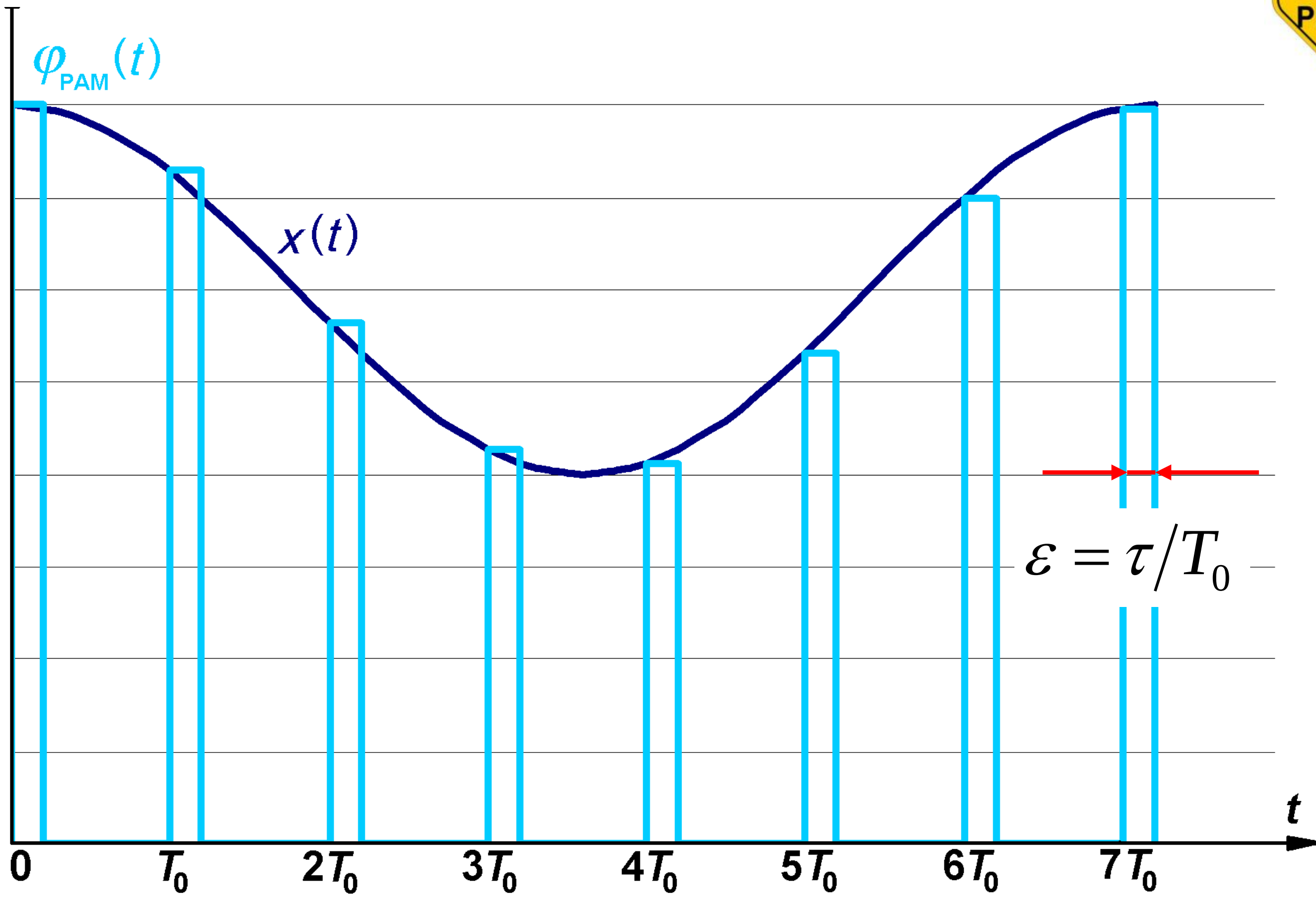
Q3. Compare the bandwidth of the signal $y(t)$ to the total bandwidth of the three signals $x_1(t), x_2(t), x_3(t)$.

Q3. Redesign the commutator (more contacts and/or another commutator frequency) to achieve a smaller bandwidth B [kHz].



Pulse Amplitude Modulation (PAM)
Instantaneous sampling

WORK IN PROGRESS



Find the Fourier transform of the PAM signal.

$$\varphi_{\mathrm{PAM}}(t) = \sum_{l=-\infty}^{\infty} x(lT_0)\,\pi_\tau(t - lT_0)$$

PAM - instantaneous sampling spectral analysis

$$\varphi_{\mathrm{PAM}}(t)=\sum_{l=-\infty}^{\infty}x(lT_0)\pi_\tau(t-lT_0)$$

$$\varphi_{\mathrm{PAM}}(t)=\sum_{l=-\infty}^{\infty}x(lT_0)\{\delta(t-lT_0)*\pi_\tau(t)\}$$

$$\varphi_{\mathrm{PAM}}(t)=\left\{\sum_{l=-\infty}^{\infty}x(lT_0)\delta(t-lT_0)\right\}*\pi_\tau(t)$$

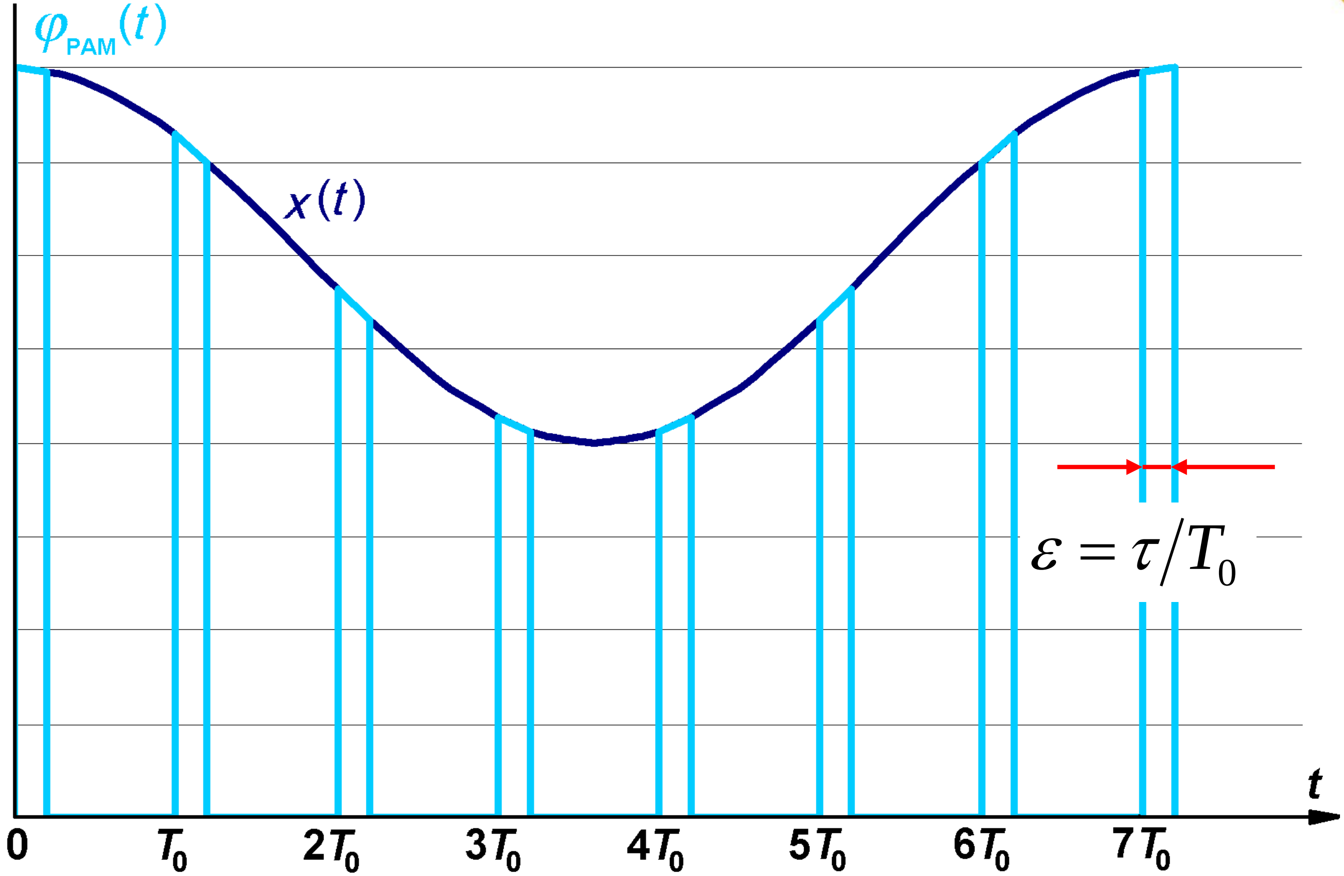
$$\varphi_{\mathrm{PAM}}(t)=\left[x(t)\delta_{T_0}(t)\right]*\pi_\tau(t)$$

$$X_{\mathrm{PAM}}(\omega)=\tau\,\mathrm{Sa}\frac{\omega\tau}{2}X_{\mathrm{s}}(\omega)=\varepsilon\mathrm{Sa}\frac{\omega\tau}{2}\sum_{l=-\infty}^{+\infty}X(\omega-l\omega_0)$$

$$\varepsilon=\tau/T_0$$

Pulse Amplitude Modulation (PAM)

Natural sampling (keying)



Find the Fourier transform of the PAM signal.

$$\varphi_{\text{PAM}}(t) = c(t)x(t) = x(t)\underbrace{\sum_{l=-\infty}^{\infty}\pi_{\tau}(t-lT_0)}_{c(t)}$$

PAM - natural sampling – spectral analysis

The carrier signal is periodic; its Fourier series is given by:

$$\boldsymbol{c(t) = 2\varepsilon \sum_{l=0}^{\infty} \mathrm{Sa}(l\pi\varepsilon)\cos(l\omega_0 t)}$$

Natural sampling is given by:

$$\boldsymbol{\varphi_{\mathrm{PAM}}(t) = c(t)x(t)}$$

$$\boldsymbol{\varphi_{\mathrm{PAM}}(t) = 2\varepsilon \sum_{l=0}^{\infty} \mathrm{Sa}(l\pi\varepsilon)\cos(l\omega_0 t)x(t)}$$

Fourier spectrum of natural sampling is given by:

$$X_{\mathrm{PAM}}(\omega) = \varepsilon \sum_{l=-\infty}^{\infty} \mathrm{Sa}(l\pi\varepsilon) X(\omega - l\omega_0)$$

$$\omega_0 = 2\pi/T_0 ; \varepsilon = \tau/T_0$$

Summary

- Several signals of practical interest does not fit the condition of limited energy to be Fourier transformable.

- The concept of the Dirac delta function makes possible to determine Fourier transforms of some signals carrying unlimited energy.

- The Dirac delta assignes to a signal its sample.

- The comb distribution (a sequence of periodically repeated Dirac deltas) is useful when modelling a signal sampling process and deriving a sampled signal spectrum.

- Signal sampling has to be performed at frequency equal to a double signal bandwidth.

- Proper signal sampling does not result in a loss of intersample signal values; an ideal lowpass filtering is sufficient for a signal reconstruction.

Set of orthogonal Sampling functions

Set of orthogonal *Sampling* functions:

$$\left\{\mathrm{Sa}\,\omega_{\mathrm{b}}(t-nT): n=0,\pm1,\pm2,\ldots;\ \omega_{\mathrm{b}}=\pi/T\right\}$$

Orthogonality over the interval (-∞, +∞):

$$\mathrm{Sa}\,\omega_{\mathrm{b}}(t-kT)\perp \mathrm{Sa}\,\omega_{\mathrm{b}}(t-nT)\Leftrightarrow$$

$$\Leftrightarrow\int_{-\infty}^{\infty}\mathrm{Sa}\,\omega_{\mathrm{b}}(t-kT)\,\mathrm{Sa}\,\omega_{\mathrm{b}}(t-nT)\,dt=0$$

The proof is based on the Rayleigh theorem:

$$\mathrm{Sa}\,\omega_{\mathrm{b}}(t-kT)\leftrightarrow\frac{\pi}{\omega_{\mathrm{b}}}\Pi_{2\omega_{\mathrm{b}}}(\omega)e^{-jkT\omega}$$

$$\mathrm{Sa}\,\omega_{\mathrm{b}}(t-nT)\leftrightarrow\frac{\pi}{\omega_{\mathrm{b}}}\Pi_{2\omega_{\mathrm{b}}}(\omega)e^{-jnT\omega}$$

Set of orthogonal Sampling functions

$$\int_{-\infty}^{\infty} \mathrm{Sa}\,\omega_{\mathrm{b}}(t-kT)\mathrm{Sa}\,\omega_{\mathrm{b}}(t-nT)dt =$$

$$= \frac{\pi}{2\omega_{\mathrm{b}}^2}\int_{-\infty}^{\infty} \Pi_{2\omega_{\mathrm{b}}}(\omega)e^{-jkT\omega}\Pi_{2\omega_{\mathrm{b}}}(\omega)e^{jnT\omega}d\omega =$$

$$= \frac{\pi}{2\omega_{\mathrm{b}}^2}\int_{-\omega_{\mathrm{b}}}^{\omega_{\mathrm{b}}} e^{j(n-k)T\omega}d\omega = T\mathrm{Sa}(n-k)\pi = \begin{cases} T, k=n \\ 0, k\neq n \end{cases}$$

The Kotielnikov-Shannon series:

$$x(t) = \sum_{-\infty}^{\infty} x(nT)\mathrm{Sa}\,\frac{\pi(t-nT)}{T} = \sum_{-\infty}^{\infty} x(nT)\mathrm{Sa}\,\omega_{\mathrm{b}}(t-nT)$$

$$\omega_{\mathrm{b}} = \pi/T$$

is a Fourier series over a set of orthogonal sampling functions; Fourier coefficients are equal to signal samples.